Stochastic Lorenz Systems are Generalized Langevin Systems

Soon Hoe Lim SOON.HOE.LIM@SU.SE
Nordita
KTH Royal Institute of Technology and Stockholm University
Stockholm 106 91, Sweden

Abstract

In this short note, we show that the x-component of a stochastic version of the famous Lorenz-63 system satisfies a generalized Langevin equation. We then give a few insightful remarks from the point of view of nonequilibrium statistical mechanics (via Kac-Zwanzig Hamiltonian formalism), study maximal transport (upper bound on time average of an observable), present a homogenization result and raise some questions for future work.

Contents

1. Introduction

We consider the following stochastic version (in Itô form) of the Lorenz-63 model:

$$\dot{x} = \sigma(y - x), \tag{1}$$

$$\dot{y} = -y + rx - xz + \sigma_y \dot{W}_y, \tag{2}$$

$$\dot{z} = -bz + xy + \sigma_z \dot{W}_z, \tag{3}$$

where $\sigma_y, \sigma_z \geq 0$, W_y and W_z are independent Wiener processes. The initial conditions and parameters (σ, r, b) are chosen so that the underlying deterministic system behaves chaotically (e.g., the classical values $(\sigma, r, b) = (10, 28, 8/3)$).

The deterministic version of the above model was first introduced in Lorenz (1963) as a simplified mathematical model for atmospheric convection, whereas the stochastic version was recently studied in Lorenz (1963); Agarwal and Wettlaufer (2016); Weady et al. (2018). This stochastic model can also be viewed as an effective model obtained in the white noise limit of a suitable open system (e.g., that described by coupling three Lorenz-63 systems with appropriate time scale separation Givon et al. (2004)).

In this note, we are interested in studying the x-dynamics. Figure 1 shows a typical trajectory of the x-component, which can be seen to oscillate between the states centered around $x = -10$ and those around $x = 10$ in a random manner. It seems then reasonable to describe the x-component

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as a kind of nonlinear stochastic oscillator. We are going to make this observation precise in the following. We will take $r > 1$ in the following.

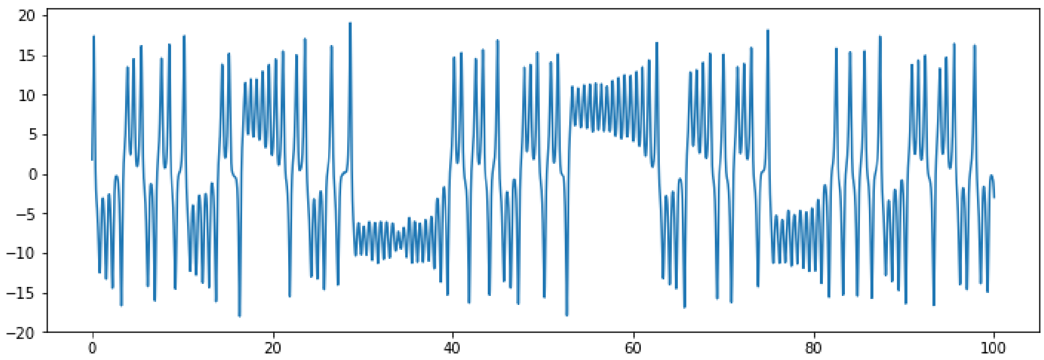

Figure 1: A realization of x-component of the stochastic Lorenz system (1)-(3), with the classical parameter values, $\sigma_y = \sigma_z = 1$, and $x(0), y(0), z(0) \sim$Uniform$[-10, 10]$.

2. From Stochastic Lorenz System to Generalized Langevin System

Using the first two equations, we derive:

$$\ddot{x} = -(\sigma+1)\dot{x} + \sigma(r-1)x - \sigma xz + \sigma\sigma_y \dot{W}_y, \tag{4}$$

and one can also show:

$$\dot{z} + bz = \frac{1}{2\sigma}\left(\frac{d}{dt}(x^2) + 2\sigma x^2\right) + \sigma_z \dot{W}_z. \tag{5}$$

We can write the solution for z as:

$$z(t) = z(0)e^{-bt} + \frac{1}{2\sigma}\int_0^t e^{-b(t-s)}\left(\frac{d}{ds}(x^2(s)) + 2\sigma x^2(s)\right)ds + \sigma_z \int_0^t e^{-b(t-s)} dW_z(s). \tag{6}$$

Performing an integration by part for the first integral on the right hand side above gives:

$$z(t) = \eta(0)e^{-bt} + \gamma\int_0^t e^{-b(t-s)}x^2(s)ds + \frac{x^2(t)}{2\sigma} + \sigma_z\int_0^t e^{-b(t-s)}dW_z(s) =: \eta(t) + \frac{x^2(t)}{2\sigma} + \xi(t), \tag{7}$$

where $\eta(0) = z(0) - \frac{x(0)^2}{2\sigma}$ and $\gamma = 1 - \frac{b}{2\sigma} > 0$. Note that η solves $\dot{\eta} = -b\eta + \gamma x^2$.

Combining what we have so far, we see that the x-component of the stochastic Lorenz equation is the solution to:

$$\dot{x} = v, \tag{8}$$

$$\dot{v} = -(\sigma+1)v + \sigma(r-1)x - \frac{x^3}{2} - \sigma x\eta + \sigma\sigma_y\dot{W}_y - \sigma x\xi, \tag{9}$$

$$\dot{\eta} = -b\eta + \gamma x^2, \tag{10}$$

$$\dot{\xi} = -b\xi + \sigma_z\dot{W}_z, \tag{11}$$

with the initial conditions $v(0) = -\sigma x(0) + \sigma y(0)$, $\xi(0) = 0$ and $\eta(0) = z(0) - \frac{x(0)^2}{2\sigma}$.

This is a *generalized version of Langevin-Kramers equation*, modeling the damped and forced motion of a particle in a double well potential. The particle is driven by a stochastic force, modeled by sum of an additive white noise and a multiplicative colored noise. The deterministic forcing term, $-\sigma x\eta$, depends on the memory of the particle past motion and is essential for emergence of the chaotic behavior. This is of course not a new result in the deterministic case (see Festa et al. (2002a,b) and the references therein) but we are not aware of any studies for the stochastic case.

It is interesting to note that the additive white noise in the z-component of the original Lorenz system leads to the multiplicative colored noise in the v-component of the Langevin system. The hopping of x between the two states, $x = \pm\sqrt{2\sigma(r-1)}$, of the Langevin system is dictated by the competition between the deterministic and stochastic forcing.

Remark 1. Had we put the additive white noise term, $\sigma_x \dot{W}_x$, in the x-equation of the original model, the extra term[1] $\sigma_x \ddot{W}_x + \sigma x\theta$, where $\dot{\theta} = -b\theta + x\dot{W}_x/\sigma$, would appear on the right hand side in the equation for v above.

Remark 2. We can compare the Langevin system to another widely studied chaotic dynamical system, the Duffing oscillator: $\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = \gamma\cos(\omega t)$. Note that the forcing in the Duffing equation is external, whereas that in the v-equation is internal. Both types of forcing give rise to chaotic behavior in certain parameter regime.

Remark 3. (Two limiting cases) Sending $\sigma \to 0$ in the Langevin system, we have:

$$\ddot{x} = -\dot{x} - x^3/2, \tag{12}$$

describing an underdamped particle in a quartic potential. On the other hand, in the limit $\sigma \to \infty$, we have

$$\dot{x} = (r-1)x - x\eta + \sigma_y \dot{W}_y - x\xi, \tag{13}$$

where $\eta(t) = z(0)e^{-bt} + \int_0^t e^{-b(t-s)}x^2(s)ds$ (i.e., it solves $\dot{\eta} = -b\eta + x^2$, with $\eta(0) = z(0)$) and ξ is the colored noise as before. Equivalently, in terms of only the x-variable:

$$\dot{x}(t) = ((r-1) - z(0)e^{-bt})x(t) - x(t)\int_0^t e^{-b(t-s)}x^2(s)ds + \sigma_y \dot{W}_y - \sigma_z x(t)\int_0^t e^{-b(t-s)}dW_z(s). \tag{14}$$

This describes an overdamped particle in a time-varying double well like potential, driven by an additive white noise and a multiplicative colored noise. Deterministic chaos and stochastic noise compete to dictate the x-dynamics as one varies the Prandtl number $\sigma > 0$.

Remark 4. In fact, the x-component of the stochastic Lorenz system belongs to the class of generalized Langevin equations (GLEs) studied in our recent paper Lim et al. (see also the references therein for a literature review of GLEs). We can write $\eta = \beta + \frac{\gamma}{b}x^2$, where β solves

$$\dot{\beta} = -b\beta - \left(\frac{2}{b} - \frac{1}{\sigma}\right)xv, \tag{15}$$

1. One needs to interpret any derivative of the Wiener process in a generalized sense.

with $\beta(0) = z(0) - x^2(0)/b$. Therefore, the x-component of the stochastic Lorenz equation solves the GLE:

$$\dot{x} = v, \tag{16}$$

$$\dot{v} = -(\sigma+1)v + \sigma(r-1)x - \frac{\sigma}{b}x^3 - \sigma x\beta + \sigma\sigma_y \dot{W}_y - \sigma x\xi, \tag{17}$$

$$\dot{\beta} = -b\beta - \left(\frac{2}{b} - \frac{1}{\sigma}\right) xv, \tag{18}$$

$$\dot{\xi} = -b\xi + \sigma_z \dot{W}_z, \tag{19}$$

where $x(0)$ is as before, $v(0) = \sigma(y(0) - x(0))$, $\xi(0) = 0$ and $\beta(0) = z(0) - x^2(0)/b$.

Equivalently, in the form of the GLEs in Lim and Wehr (2018):

$$\dot{v}(t) = -(\sigma+1)v(t) + \sigma F(t, x(t)) + \sigma x(t) \int_0^t e^{-b(t-s)} \left(\frac{2}{b} - \frac{1}{\sigma}\right) x(s)v(s)ds \tag{20}$$

$$+ \sigma\sigma_y \dot{W}_y + \sigma x(t)\xi(t), \tag{21}$$

where $F(t,x) = (r - 1 - \beta(0)e^{-bt} - \sigma\xi(0)e^{-bt})x - \frac{1}{b}x^3 = -U'(t,x)$, where U is a time-varying double well potential (c.f. Eqn. (2.2) in Lim et al.),), $\xi(t) = \xi(0)e^{-bt} - \sigma_z \int_0^t e^{-b(t-s)} dW_z(s)$, with $\xi(0) \sim N(0, \sigma_z^2/2b)$. If we send $\sigma \to \infty$ in the above, then:

$$\dot{x}(t) = F(t, x(t)) + \frac{2}{b}x(t) \int_0^t e^{-b(t-s)} x(s)\dot{x}(s)ds + \sigma_y \dot{W}_y - \sigma_z x(t)\xi(t). \tag{22}$$

Looking at the stochastic Lorenz-63 system through the lens of GLE could give us some refreshing perspective on the chaotic and stochastic behavior of x-component of the system. In particular, the x-dynamics of the stochastic Lorenz-63 system can be derived from a Kac-Zwanzig Hamiltonian model, describing an anharmonic oscillator interacting with an equilibrium heat bath (sum of two independent Markovian and non-Markovian heat bath at two different temperatures) and is subject to a deterministic time-dependent external force f Zwanzig (1973); Lim and Wehr (2018):

$$H(x, v, \{x\}_k, \{v\}_k) = \frac{1}{2}v^2 + U(x) + f(t)\frac{x^2}{2} + \sum_{k1} \left(\frac{1}{2}v_{k1}^2 + \frac{1}{2}\omega_{k1}^2 \left(x_{k1} - \frac{c_{k1}}{\omega_{k1}^2}\phi_1(x)\right)^2\right) \tag{23}$$

$$+ \sum_{k2} \left(\frac{1}{2}v_{k2}^2 + \frac{1}{2}\omega_{k2}^2 \left(x_{k2} - \frac{c_{k2}}{\omega_{k2}^2}\phi_2(x)\right)^2\right), \tag{24}$$

with $\phi_1(x) = x$, $\phi_2(x) = \sigma x^2/2$ (quadratic coupling), $f(t) = \sigma(\beta(0) + \sigma\xi(0))e^{-bt}$, $U(x) = +\sigma x^4/4b - \sigma(r-1)x^2/2$, the c_{ki}, ω_{ki} $(i = 1, 2)$ are appropriate variables determining the damping and noise term, and the initial data $x_{ki}(0)$ and $v_{ki}(0)$ are independent mean zero Gaussian random variables with covariance proportional to the temperature of the respective heat bath (i.e., distributed according to a Gibbs measure). By relating σ_y and σ_z to the temperature of the respective heat bath, fluctuation-dissipation relation of the first kind and second kind are automatically satisfied respectively.

On the other hand, in general the memory term in a GLE (not restricted to the one in Remark 4) may be responsible for any chaotic behavior of the system. This also justifies the use of GLEs to model physical processes in climate science Gottwald et al. (2017).

3. Maximal Transport

Recall that x is as before, $v = \sigma(y - x)$, and $\eta = z - x^2/(2\sigma)$, so $xy = xv/\sigma + x^2$. Let $\langle x \rangle_T := \frac{1}{T}\int_0^T x(t)dt$. The following results are obtained by extending the background method in Souza and Doering (2015) to the stochastic case.

Proposition 1 *Let $T > 0$, then a.s.,*

$$\langle x^2\rangle_T = \langle xy\rangle_T + O(1/T), \tag{25}$$

$$\langle y^2\rangle_T = r\langle xy\rangle_T + \sigma_y\langle y\dot{W}_y\rangle_T - b\langle z^2\rangle_T + \sigma_z\langle z\dot{W}_z\rangle_T + \frac{\sigma_y^2+\sigma_z^2}{2} + O(1/T), \tag{26}$$

$$\langle xy\rangle_T = b(r-1) + \frac{\sigma_z\langle z\dot{W}_z\rangle_T}{r-1} - 2\sigma_z\langle \dot{W}_z\rangle_T + \frac{\sigma_y}{r-1}\langle y\dot{W}_y\rangle_T + \frac{\sigma_y^2+\sigma_z^2}{2(r-1)} \tag{27}$$

$$- \frac{1}{(r-1)}\left\langle (x-y)^2 + b(z-r+1)^2\right\rangle_T + O(1/T), \tag{28}$$

as $T \to \infty$.

Theorem 3.1

$$\limsup_{T\to\infty}\langle x^2\rangle_T \leq x_{st}^2 + \frac{\sigma_y^2+\sigma_z^2}{2(r-1)} + \frac{1}{r-1}\limsup_{T\to\infty}(\sigma_z\langle z\dot{W}_z\rangle_T + \sigma_y\langle y\dot{W}_y\rangle_T), \tag{29}$$

$$\limsup_{T\to\infty}\langle y^2\rangle_T \leq y_{st}^2 + \frac{r(\sigma_y^2+\sigma_z^2)}{2(r-1)} + \frac{2r-1}{r-1}\limsup_{T\to\infty}(\sigma_z\langle z\dot{W}_z\rangle_T + \sigma_y\langle y\dot{W}_y\rangle_T), \tag{30}$$

$$\limsup_{T\to\infty}\langle xy\rangle_T \leq x_{st}y_{st} + \frac{\sigma_y^2+\sigma_z^2}{2(r-1)} + \frac{1}{r-1}\limsup_{T\to\infty}(\sigma_z\langle z\dot{W}_z\rangle_T + \sigma_y\langle y\dot{W}_y\rangle_T), \tag{31}$$

a.s., where x_{st} and y_{st} are the steady-state solution of the deterministic Lorenz-63 equations, i.e., $x_{st} = y_{st} = \pm\sqrt{b(r-1)}$.

Proof This follows from the previous proposition, and the fact that $\langle \dot{W}\rangle_T/T = W_T/T \to 0$ as $T \to \infty$ a.s. for any Wiener process W. ■

Taking the ensemble average, we find:

$$\mathbb{E}\limsup_{T\to\infty}\langle xy\rangle_T \leq x_{st}y_{st} + \frac{\sigma_y^2+\sigma_z^2}{2(r-1)}. \tag{32}$$

As expected, this stochastic upper bound is larger than the deterministic counterpart of Souza and Doering Souza and Doering (2015), in agreement with Agarwal and Wettlaufer (2016). However, the above derived bound is smaller than the one in Agarwal and Wettlaufer (2016), where the x-equation is driven by a white noise there. Following Agarwal and Wettlaufer (2016), one can study the effect of noise on the maximal transport as the Rayleigh number r is varied. A natural question is how does the bound relate to the averages with respect to the invariant measure of the stochastic Lorenz system and is it tight (it is saturated by steady convection solutions in the deterministic case). It will be good to investigate this in the GLE framework. One can also try to extend the optimal control method of Souza and Doering to obtain a tight bound for the stochastic case.

4. A Homogenization Result

We are interested in the **joint limit** of $\sigma, b, \sigma_z \to \infty$ at the same rate of the GLE in Remark 4 (or equivalently the original stochastic Lorenz system). We replace $\sigma \mapsto \sigma_0/\epsilon$, $b \mapsto b_0/\epsilon$, $\sigma_z \mapsto \tilde{\sigma}_z/\epsilon$ in the GLE, where $b_0, \sigma_0, \tilde{\sigma}_z > 0$ are proportionality constants and $\epsilon > 0$ is a small parameter. We then study the limit $\epsilon \to 0$ of the resulting rescaled GLE by formally applying the result in Lim et al..

A Homogenization Result. In the limit $\epsilon \to 0$ the x-component, x^ϵ, of the family of the rescaled GLEs (or the rescaled stochastic Lorenz equations) converge to X, satisfying the linear Itô SDE:

$$dX_t = (r-1)X_t dt + S(X_t)dt + \sigma_y dW_y - \frac{\tilde{\sigma}_z}{b_0} X_t dW_z, \tag{33}$$

where $r > 1$, $\sigma_y \geq 0$, and $S(X) = RX$, with $R = \sigma_0 \tilde{\sigma}_z^2/(2b_0^2(b_0+\sigma_0)) > 0$, is a noise-induced drift. More precisely, with more work[2] one may show that $\sup_{t\in[0,T]} |x_t^\epsilon - X_t| \to 0$ in probability, for any $T > 0$.

Taking the above limit naively (i.e., by simply setting $\epsilon = 0$ in the prelimit equations and then rearrange the terms[3]) would incorrectly give us the above Itô SDE without the noise-induced drift term. This noise-induced drift effectively further destabilizes the limiting system. Its presence is due to, or can be traced back to, the presence of the multiplicative noise term, $\sigma_0 x^\epsilon \xi^\epsilon$, in the prelimit equations, or equivalently, the additive white noise term in the z-equation in (3). Note that when $\sigma_y = 0$, X is simply a geometric Brownian motion (Black-Scholes model).

It would be interesting to investigate a generalized version of the stochastic Lorenz system and study its (anomalous) diffusive behavior. Also, one could study stochastic thermodynamics and fluctuation theorems. We leave these to future work.

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3. Or, by first taking $\sigma \to 0$ and then the other two limits.

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